# The moduli stack of elliptic curves

Functor of points!

$$\begin{tikzcd} \text{Sch} \arrow[r, hook, "\text{よ}"] \arrow[dr, bend right] & {[\text{Aff}^{op}, \text{Set}]} \arrow[d] \\ & {[\text{Sch}^{op}, \text{Set}]} \end{tikzcd}$$

Elements of $X(R)$ are called $R$-points, because...

Great for constructing morphisms. Less good for constructing objects: no general sufficient conditions, but many necessary ones, for example various sheaf conditions.

To specify a map to $X$, enough to specify it on an open Zariski cover. To encode this, let (say) $U, V$ be a Zariski open cover of $X$, and let $F: \text{Sch}^{op} \to \text{Set}$. Then Zariski descent for this cover means

$$E = U \amalg V \quad E \to X$$

$$E \times_X E \rightrightarrows E \qquad \text{(reflexive coeq / 1-truncated simplicial diagram)}$$

$$F(X) \xrightarrow{\sim} \lim \left(F(E) \rightrightarrows F(E \times_X E)\right)$$

We'll come back to this later.

An elliptic curve $E/S$ is a smooth, proper alg curve of genus 1 (i.e. $\Omega^1_{E/S}$ is trivial) with a distinguished $S$-point. $S$ will often be a DVR or field.

This is a group scheme:

- Can embed $E$ in $\mathbb{P}^2$, as a cubic. Any line $L \subseteq \mathbb{P}^2$ intersects $E$ (with mult) at three points $P_1, P_2, P_3$. Say $P_1 + P_2 + P_3 = 0$.

- Over $\mathbb{C}$, $E$ is (analytically) a complex torus. Usually torus addition.

- $E$ parametrises line bundles over itself:

$$E(R) \cong \{\text{line bundles on } E \times_S R \text{ Spec} R\}/\sim.$$

If $R = k$ a field (for example) $\sim$ is just isomorphism. The group law on $E$ corresponds to $\otimes$.

Seek a space $M_{ell}$ parametrising ell curves. First work over $\mathbb{C}$. ~~An ell curve is~~

So that $M_{ell}(R) = \{(\text{families of}) \text{ ell curves}/R\}/\text{iso}.$

An EC is given by a lattice $\langle 1, \tau \rangle$, where we may take $\tau \in H$, the upper half plane. $\tau_1, \tau_2$ give isomorphic lattices $\iff \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \tau_1 \\ 1 \end{pmatrix} \cong \begin{pmatrix} \tau_2 \\ 1 \end{pmatrix}$ for some $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2\mathbb{Z} \iff \tau_2 = \frac{a\tau_1 + b}{c\tau_1 + d}$

So we might try

$$M_{ell} := H/SL_2(\mathbb{Z})$$

Assuming this is algebraic, $M_{ell}(\mathbb{C}) = \{EC/\mathbb{C}\}/\text{iso}.$

So this is a good first approximation, but can such an approach ever work? No, due to isotrivial families.

~~Let~~ Let $S = \operatorname{Spec} k$, char $k \neq 2$. ~~Then~~ Fix an ell curve $E$. Let

$X = \mathbb{G}_m \times E$ $\qquad \mathbb{Z}/2 \circlearrowright X$ by $(u \mapsto -u)$ on $\mathbb{G}_m$
$(e \mapsto -e)$ on $E$

$X/(\mathbb{Z}/2)$ is another $E$-bundle over $\mathbb{G}_m$, in that all fibres are iso to $E$. So the induced map $\mathbb{G}_m \to M_{ell}$ is constant. But $X/\mathbb{Z}/2 \not\cong \mathbb{G}_m \times E$. How to fix this? If $M_{ell}$ is to be "just a scheme", there's really no way. We must look in the mirror and define an object by the functor it represents, and the functor $M_{ell}$ represents should be groupoid-valued, to acct for isomorphisms.

~~Sheaf of groupoids - homotopy descent.~~

Etale maps, étale descent

So that's a stack - a groupoid-valued functor $\mathrm{Aff}^{op} \to \mathrm{Grd}$ w/ sheaf condition (mention)

You might think these things would be hard to work with, but one thing is clear: how to define limits.

In fact ~~in many cases~~ for many nice stacks $M$ there is a morphism $X \to M$ for $X$ a scheme such that for any other mor from a scheme $Y \to M$, $X \times_M Y$ is a scheme. e.g.

moduli space of ell curves with level 5 structure, $M_{ell}[5]$. No automorphism preserves this, ~~so~~ and compact 1-d complex manifold, so alg curve.

$$\begin{tikzcd} \mathbb{G}_m \arrow[r, "\mathbb{G}_m \times E"] \arrow[d, "r \mapsto r^2"'] & M_{ell}[5] \arrow[d] \\ \mathbb{G}_m \arrow[r, "(\mathbb{G}_m \times E)/\mathbb{Z}/2", "\text{exists.}"'] & M_{ell} \end{tikzcd}$$

Noncontractible loop.

A really down-to-earth presentation of $M_{ell}$: any elliptic curve can be written as

(suppose for simplicity char $k \neq 2$)

affine eq $\quad y^2 = x^3 + ax + b$ (projective completion has 1 extra pt, which we take to be the identity)

ell curve if $\Delta := -16(4a^3 + 27b^2) \neq 0$

$(y^2 = x^3 + ax + b) \cong (y^2 = x^3 + a'x + b')$

$\iff \exists\, u \in k^\times \quad a' = u^4 a, \ b' = u^6 b$

So ~~the moduli stack of~~ $M_{ell}$ can be given by

$$\operatorname{Spec} R \longmapsto \begin{cases} \text{objects} & (a,b) \in R^2 \quad \Delta(a,b) \neq 0 \\ \text{morphisms} & (a,b) \to (u^4 a, u^6 b) \quad \forall\, u \in R^\times. \end{cases}$$

This presentation is very convenient for calculations.

Note: If $j = \frac{-1728(4A)^3}{\Delta}$ then $j$ is an iso invariant.

In fact, if ~~j~~ $r \in R$ there is ~~R alg closed~~ (unique up to alg closed iso) $E/R$ with $j(E) = r$ $\therefore$ "up to stackiness", over ACF

$$j : M_{ell} \xrightarrow{\sim} \mathbb{A}^1.$$

We have the universal ell curve

$$\begin{array}{ccc} \mathcal{E} & \longrightarrow & \overline{\mathcal{E}} \\ \downarrow & \lrcorner & \downarrow \\ M_{ell} & \longrightarrow & \overline{M_{ell}} \end{array}$$

Can compactify by adding nodal cubics, which are some curves with $\Delta = 0$. ~~All iso~~ All iso over ACF, so $\cong \mathbb{P}_1 / \{0, \infty\}$

e.g. $y^2 = x^3 + x^2$

$$j : \overline{M_{ell}} \xrightarrow{\sim} \mathbb{P}^1.$$ Extra point called the cusp.

Change of tack: what's a 1-form on $M_{ell}$? ~~If~~ ~~Must be an~~ ~~If we wr~~ Let's work over $\mathbb{C}$.

If we think of $M_{ell}$ as $\mathcal{H} // SL_2(\mathbb{Z})$, then this must be a $SL_2(\mathbb{Z})$-equivariant 1-form on $\mathcal{H}$, i.e.

$$\omega \in \Omega^1(\mathcal{H}) \quad g^* \omega = \omega \quad \text{all } g \in SL_2(\mathbb{Z})$$

(do calc) If $\omega = f(z)\,dz$, then

$$f\left(\frac{az+b}{cz+d}\right) = \frac{1}{(cz+d)^2} f(z), \quad \text{all } z \in \mathcal{H}$$

Such an $f$ is a modular form of weight **2** if it extends to the cusp.

(This is iff $f$ has a limit as $z \to$ vertical $\infty$).

So a modular form of weight 2 is a section of $\Omega^1(\overline{\mathcal{M}}_{ell})$.

Note: $\begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$.

So what does it look like?